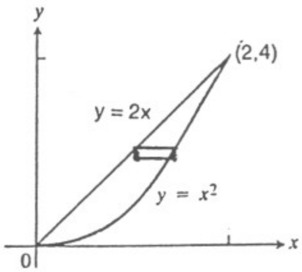
FINDING VOLUMES BY INTEGRATION

DISK METHOD

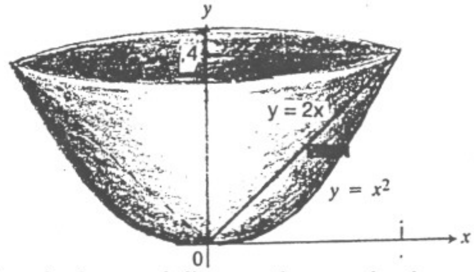
Overview

There are two commonly used ways to compute the volume of a solid -- the ***Disk Method*** and the ***Shell Method***. Both involve slicing the volume into small pieces, finding the volume of a typical piece, and then adding up all the little pieces to form a Riemann Sum.

Consider the region in the first quadrant bounded by the curves $y = 2x$ and $y = x^2$. By solving these equations simultaneously we can see that the two curves intersect at the points (0,0) and (2,4). Suppose we wish to compute the volume of the solid formed when this region is rotated about the Y-axis.



Region to be rotated



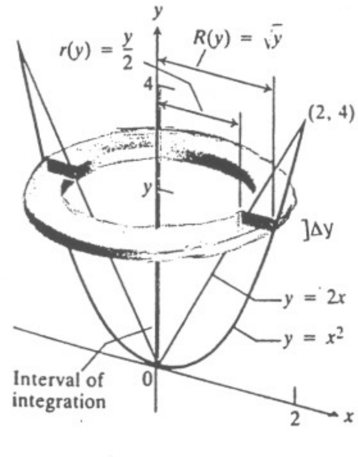
Bowl-shaped figure formed when region is rotated about the Y-axis

This handout sheet will only discuss the Disk Method, but there is another handout which computes the volume of this same solid by using the Shell Method.

Computing the Volume of One Disk

Imagine that the "bowl" is sliced up by a set of planes perpendicular to the axis of rotation -- the Y-axis in this case. Each slice can be approximated by a shape which looks like a washer. (You can also think of each slice as looking like a slice of pineapple or a coin with a hole drilled through the center of it.) A typical such washer is shown below.

We must compute the volume of one such washer. If there were no hole in the middle, then the slice would be a thin cylinder, called a disk. We will compute the volume of the washer by taking the volume of the disk and then subtracting the volume of the hole.



The volume of a cylinder is $\pi r^2 h$. The radius of the disk extends from the Y-axis to the curve $y = x^2$. We will denote this radius as x.

The height of the disk is the distance between the horizontal slicing planes, which we will denote as Δy. The volume of the disk is therefore $\pi x^2 \Delta y$.

We want to express everything in terms of the same variable. We know that the radius of the disk extends to the curve $y = x^2$. Since we are in the first quadrant, we can rewrite this equation as $x = \sqrt{y}$. Substituting for x, the volume of the disk can be expressed as $\pi (\sqrt{y})^2 \Delta y$, which can be simplified to $\pi y \Delta y$.

Next we must compute the volume of the hole. The hole is also a cylinder with height Δy and its radius extends from the Y-axis to the line $y = 2x$. We will denote its radius as x, where x must satisfy the equation $y = 2x$. The volume of the hole is therefore $\pi x^2 \Delta y$. In this case $x = \frac{y}{2}$, and so the volume of the hole could be expressed as $\pi (\frac{y}{2})^2 \Delta y$.

Now that we have expressed the volume of the disk and the volume of the hole in terms of only the one variable y, we can compute the volume of the washer by finding the difference of the two volumes. Therefore, the volume of the washer is $\pi y \Delta y - \pi (\frac{y}{2})^2 \Delta y$, which can also be written as $\pi [y - (\frac{y}{2})^2] \Delta y$.

Volume of the Entire Solid

We must now add up the volumes of all the washers. We divided the interval from $y = 0$ to $y = 4$ into segments, each of length Δy. By adding up all the washers from $y = 0$ to $y = 4$, and letting Δy approach 0, we can express the volume of the entire bowl-shaped solid by the following integral.

$$\Pi \int_0^4 y - \left(\frac{y}{2}\right)^2 dy.$$

By evaluating this integral, we can conclude that the volume of the solid is $\frac{8\Pi}{3}$.