Finding Equations of Lines

Why do we need to be able to do this?

Lines are fundamental to people and the things we create. We love shapes with straight-line borders. In science and mathematics, we love lines and their equations because they are simple to work with, yet contain powerful information.

What is the equation of a line?

The equation of a line is a linear equation in two variables. The equation of a line is an equation whose solution set is all the points of a line.

How do you recognize the equation of a line?

The equation of a line has (at most) two variables (often x and y), raised only to the first power, and not multiplied or divided by each other. The equation of a line can always be algebraically maneuvered into standard form, $Ax + By = C$, where A, B, and C are real numbers.

What should you be able to do?

Recognize the many ways equations of a line can look. There are several different forms of an equation of a line that you might encounter.

> **Slope-Intercept form, $y = mx + b$**: This is the favorite of most students. The form is easy to remember. You can read the slope m and y-intercept b right off the equation. If you don't have the y-intercept, you will have to do some algebra to use this form.

> **Point-Slope form, $y - y_1 = m(x - x_1)$:** This is my favorite form. The slope m is visible, and (x_1, y_1) is some known point on the line. I like this form the best because there is no algebra required – just plop the slope and one point into place and you're done.

> **Standard form, $Ax + By = C$:** This form is useful for comparing different types of equations. But it's not a very helpful form for graphing or writing the equation of a line. You have to do algebra to find either the slope or any point on the line.

Understand slope. Slope is a number that tells you which direction the line slants. If the slope is positive, the line slants uphill as you read from left to right. If the slope is negative, the line slants downhill. Horizontal lines have a slope of zero. The closer the slope is to zero, the closer the line is to horizontal. The further the slope is from zero,

either positive or negative, the steeper the line is. Vertical lines have undefined slope – because the "run" in the rise-over-run calculation is zero.

You can calculate slope using any two points on the line and that familiar rise-over-run calculation. Parallel lines have the same slope. Perpendicular lines have negative reciprocal slopes (that is, their slopes multiply to make –1).

If the equation of the line is in slope-intercept form or point-slope form, you can read the slope directly off the equation.

Slope is a rate of change. The units of slope are fractional, *y*-units over *x*-units, like miles per hour or dollars per day. In an application problem, look for the fractional units to help you find the slope.

Recognize linear growth, no matter how the information is given to you. Remember that there are four ways quantitative information can be presented.

Numerically: Linear functions have a constant change in *y* for every constant change in *x*. This reflects the graphical idea of a linear function – the change in *y* over the change in *x*, or $\Delta y/\Delta x$, is the constant slope of the line. One way to recognize a line is if you see a constant slope in a table of numbers.

Algebraically: The formula for a linear function can always be algebraically maneuvered into one of the common forms given above. I like to put equations into the *slope-intercept* form, $y = mx + b$. The constant *m* is the slope of the line, rise/run, $\Delta y/\Delta x$. The constant *b* is the *y*-intercept of the line, the *y*-value when $x = 0$.

The actual letters you use are not important. In fact, I like to use letters that make sense with the problem; that helps me remember what I'm looking at. The important part is – the output is a constant slope times the input, plus the *y*-intercept.

Graphically: Linear functions are the ones whose graphs are straight lines. To find the equation of the line, you measure its slope (rise over run) and read its *y*-intercept (where the line crosses the *y*-axis).

In English: Linear functions have a constant rate of change. You can often recognize the slope by its units; they are fractional units, rise/run units, like miles per hour, or dollars per pound, or people per year. The *y*-intercept is like the fixed cost or the overhead – how much *y* you have when *x* is zero.

Write an equation of a line.
All you need in order to write the equation of a line is the slope and one point. The slope might be given to you (look for fractional units!), or you might compute it from two points, or perhaps get it from another line that is parallel or perpendicular to it. The one

point is usually given to you, or you could need to find the intersection of some curves to get the point. It's usually easiest to use point-slope form or slope-intercept form when you're writing the equation of a line.

Example: The cost of renting a car is a flat $25 plus an additional 10 cents per mile. Write an equation expressing the cost of renting the car for any number of miles.

The "10 cents per mile" is a constant rate of change, so we will be writing the equation of a line.

Let's let M be the number of miles driven and R be the cost of renting the car in dollars. The slope is given by our constant 10 cents/1 mile = 0.10 dollars per mile. The $25 is the R-intercept – you can tell that because it's the amount you'd pay if you drove $M = 0$ miles. The formula would be

$$R = 25 + 0.10M.$$

You can use the formula to quickly compute the cost of renting the car and driving any amount of miles. If you drive 153 miles, it will cost you
$$R = 25 + (0.10)(153) = 40.30 \text{ dollars.}$$

Worked Examples

Example: Find the equation of the line that passes through the two points (3, 5) and (–2, 10).

In order to write the equation of a line, we need the slope and one point. Here, we can use the two points to compute the slope, and then we can use either point to write the equation.

The slope is rise/run, or $\Delta y / \Delta x$:
$$m = \frac{10 - 5}{-2 - 3} = \frac{5}{-5} = -1.$$
Then we can plug the pieces into point-slope form to write the equation:
$$y - 5 = -1(x - 3).$$
You could use the other point instead, or you could do some algebra to put this line in slope-intercept form – the equations would look different, but they would all represent the same line.

Example: A faucet is dripping water at a constant rate into a bowl. At 1:00, there was ½ cup of water in the bowl. At 1:45, there was ¾ cup of water in the bowl. How much water will be in the bowl at 3:30?

This is a linear function, because the faucet is dripping at a constant rate. The domain is the set of times (hours past noon). The range is the set of volumes in cups (numbers ≥ 0). Let t be the time, measured in hours past noon, and let W be the amount of water in

the bowl, measured in cups. There are two points given: when $t = 1$, $W = 0.5$, and when $t = 1.75$, $W = 0.75$. The slope is rise/run, $\Delta W / \Delta t = \dfrac{(0.75 - 0.5)}{(1.75 - 1)} = \dfrac{.25}{.75} = \dfrac{1}{3}$ cups per hour.

So the equation will be

$$W = \frac{1}{3}t + b.$$

To find the W-intercept, just plug in one of the points you know and solve for b:

$$\frac{1}{2} = \frac{1}{3} \cdot 1 + b, \text{ or } b = \frac{1}{6}.$$

The function that tells us how much water is in the bowl after t hours is given by

$$W = \frac{1}{3}t + \frac{1}{6}.$$

As a check, let's make sure this gives us the right answer at the other known point – if I plug in $t = 1.75$, I get $W = 0.75$, which is right. At 3:30, $t = 3.5$, and $W = 4/3$ cup.

Example: Here are two equations of lines. Do they represent parallel lines?
$$y = 2x - 4 \text{ and } 4x - 2y = 19$$

First, let's confirm that these really are both equations of lines – yes, both equations are linear equations with two variables. The first equation is in slope-intercept form, and the second one is in standard form.

Parallel lines have the same slope. So let's find the slope of each of these.

Since $y = 2x - 4$ is in slope-intercept form, we can just read the slope right off – the slope of this line is 2.

Since $4x - 2y = 19$ is in standard form, we can't read the slope directly. But we can do a small amount of algebra to put it in slope-intercept form, and then we can read the slope:

$$4x - 2y = 19$$
$$-2y = -4x + 19$$
$$y = \frac{(-4x + 19)}{-2} = 2x - \frac{19}{2}.$$

The slope of this line is also 2, so the lines have the same slope and they are parallel. (They are not the same line, because the two y-intercepts are different.)

Practice problems

1. Find the equation of the line that passes through the two points (a, b) and (c, d).

2. Find the equation that passes through (0, a) and is parallel to the line $y = mx + b$

3. A telephone plan costs $m each month plus $.15 per minute for the calls. Find the cost for a month during which you make M minutes of calls.

This represents a linear function – the cost per minute (rate of change) is a constant. So let's find the equation of the line. We'll let t be the number of minutes, and C will be the monthly cost.

We need the slope and a point. The slope is the rate of change, so the slope is .15.

The flat fee each month represents the y-intercept, how much you pay if you make no calls, or the C value if $t = 0$. The y-intercept is m.

So the equation of the line is $C = .15t + m$

Now we can use the equation to find the cost when $t = M$.

4. Find the equation of the line shown in this graph (check WAMAP)

If you know the function is linear, two points are enough to write the formula. Use the two points to find the slope, and then solve to find the *y*-intercept.

You now know almost everything there is to know about lines. This material is very important – not just throughout this course, but in calculus also. You should know:

What is a line?

A line is a fundamental geometric object. It is straight, infinitely long, and has zero width. When we draw a line on paper, the pencil mark is not really a line because it has width. Lines can exist in the plane, or in space, or in many dimensions. Here, we concentrate on lines in a plane.

The solution set for a linear equation in two variables is a set of ordered pairs that make the equation true. Just as in the case of linear equations in one variable, there are three possibilities: The equation could have no solution, so the solution set is the empty set (If the equation is equivalent to -5 = 0, say). The equation could always be true, so the solution set is the set of all ordered pairs (if the equation is equivalent to 0 = 0, say). Or the equation could be true for the set of ordered pairs that make up a line.