

Completing the square

Why do we need to be able to do this?

Completing the square is typically introduced as a way to solve quadratic equations. There are several better ways to solve quadratic equations, and most students quickly forget the method. However, completing the square continues to come up. For example, the method is needed for finding equations of circles and ellipses.

What should you be able to do?

Complete the square. The idea is to rewrite a quadratic as a perfect square plus some number. It's essentially unwinding the FOIL operation for a perfect square:

(If animated, expand then unwind these. Otherwise, just display with colors)

$$(A + B)^2 = A^2 + 2AB + B^2$$

$$(x + 3)^2 = x^2 + 2x3 + 3^2 = x^2 + 6x + 9$$

$$(t - 5)^2 = t^2 - 2t5 + 5^2 = t^2 - 10t + 25$$

$$a^2 - 14a + 49 = (a - 7)^2$$

The first two terms in this quadratic expression tell us what the square should be. The first term of the quadratic is the square of the first term of the binomial we are squaring. You can see from the FOIL pattern that the second term of the quadratic is just twice the product of the two terms in our binomial. The last step is to add whatever constant we need to make the two expressions equal.

Recipe for completing the square:

Given a quadratic, follow this recipe to complete the square:

1. Factor out any coefficient of the square term. Keep track of it, so you can include it at the end.
2. Now your quadratic has leading coefficient = 1, so it looks like $x^2 + bx + c$. The square you want will be $\left(x + \frac{b}{2}\right)^2$. It is helpful to expand this so you can see the constant term from the square.
3. Add and subtract the constant term from the square and regroup.

Example:

Complete the square: $y^2 - 6y + 15$

This has leading coefficient = 1 already, so we don't have to factor anything out. The first term (y^2) tells us that we're looking for something like $(y + a)^2$. The coefficient of the y , -6 , tells us that $a = -3$. $(y - 3)^2 = y^2 - 6y + 9$, so we'll add and subtract 9 and regroup:

$$y^2 - 6y + 15 = y^2 - 6y + (9 - 9) + 15 = (y^2 - 6y + 9) + 6, \text{ so}$$

$$y^2 - 6y + 15 = (y - 3)^2 + 6.$$

That's completing the square.