Dimensional Analysis

Link to measurement and back

Why do we need to be able to do this?

You may never have thought much about units before, except maybe when you were studying the metric system in school. In many math classes, the units are all but ignored – when you're done solving a problem, you get a number. It doesn't seem important whether that number represents the number of pounds of coffee or the speed of the bicycle in feet per second or the temperature of the water in degrees Celcius.

But the units are very important.

Here's a secret that every physicist knows but most math teachers never tell you:

The units will tell you how to solve your problem.

What should you be able to do?

Before you start playing with the numbers, look at the units. Then set up an equation with just the units. This equation is your pattern for solving the problem.

Example: Suppose you're driving to work 4 miles away. You know that it takes 10 minutes to get there (if traffic isn't too bad). What's your average speed in miles per hour?

Solution: Before you start playing with the numbers, look at the units. You have miles, minutes, and you want miles per hour. (The English word "per" means "divided by" in mathematics.) Minutes and hours measure the same attribute, time, so you'll need to know how those units relate to each other.

Start on the left with what you want (miles per hour), and then fit the pieces into their place, making sure that the units in the numerators and denominators match:

$$\frac{miles}{hour} = \frac{miles}{\min} \cdot \frac{\min}{hour}$$

Notice how the numerators and denominators "cancel," just like in any other fraction. The min/hour expression will be how many minutes per hour there are.

This equation is your pattern for solving the problem. Now you're ready to plug in the numbers.

$$x\,\frac{miles}{hour} = \frac{4\ miles}{10\ \text{min}} \cdot \frac{60\ \text{min}}{1\ hour} = 24\,\frac{miles}{hour}$$

Your average speed was 24 miles per hour.

That was a fairly easy one. This one's trickier:

Example 2: You know that your car gets 32 miles per gallon on the freeway. You've been driving at a constant 65 miles per hour for the last 2 hours. How much gasoline did you use?

Solution: First, the equation with just the units

$$gallons = \frac{gallons}{miles} \cdot \frac{miles}{hour} \cdot hour$$

Notice how I had to use gallons per mile in my unit equation so that the gallons would be in the numerator when I was done and the miles would cancel. When I plug in the numbers, I'll just be sure to put the 32 next to the miles and the 1 next to the gallons.

Then fill in the numbers:

$$x\ gallons = \frac{1\ gallon}{32\ miles} \cdot \frac{65\ miles}{1\ hour} \cdot 2\ hours = 4.0625\ gallons$$

You used 4.0625 gallons. Can we check? OK, 2 hours at 65 miles per hour is 130 miles. Each gallon will go 32 miles, so yes – we should need a bit more than 4 gallons to go 130 miles.

Using the units to set up (or check) your problem this way is called *dimensional analysis*. Every physics student learns about this right away, but math students often don't.

Worked examples:

Example: My brand of laundry detergent comes in a 128-ounce package. On the label, it says it contains enough detergent for 88 loads. Unfortunately, I have lost the little cup that I'm supposed to use to measure the detergent, and I'll have to use an ordinary measuring cup. How much laundry detergent should I use per load?

Ounces per load tells you how to set up the problem: 128 ounces divided by 88 loads gives 16/11 ounces per load. Check – well, 128 is between 1 and 2 times 88, so I expected a number between 1 and 2. This seems right.

Reality check: I can't measure 16/11 ounces with my measuring cup! My measuring cup shows common fractions of a cup. How much of a cup is 16/11 ounces?

There are 8 ounces in a cup (8 ounces per cup). Let the units tell you what to do. I have ounces, and I want to end up with cups, so I need to multiply my ounces by cups per ounce:

$$\frac{16}{11} ounces \cdot \frac{1\ cup}{8\ ounces} = \frac{2}{11} cups$$

I'll need 2/11 cups of detergent per load.

Second reality check: My measuring cup doesn't show 11ths of a cup. It shows ¼, 1/3,1/2, and so on. What should I really do?

This is where the calculator approximation can come in handy. The calculator says that I need about 0.181818 cups. The smallest mark on my measuring cup is ¼ cup, which is 0.25 cups. So I will just fill my measuring cup to a little less than the ¼ cup mark, and hope that's good enough. And next time, I'll make sure not to lose the little cup that comes with the detergent.

Example: I bought a box of Chocolate Frosted Sugar Bombs Cereal. The price for the 14-ounce box was $2.99. What was the price of the cereal per gram?

First, we need to look up the conversion between grams and ounces – one ounce is about 28.35 grams. Then we can let the units tell us what to do. We want dollars per gram. We'll set it up without the numbers in place first to be sure that the units come out right, then put in the pieces.

$$\frac{dollars}{ounce} \cdot \frac{ounces}{gram} = \frac{dollars}{gram}$$

$$\frac{2.99\ dollars}{14\ ounces} \cdot \frac{1\ ounces}{28.35\ gram} \cong 0.0075\ \frac{dollars}{gram}$$

It cost about ¾ of a cent per gram. Does that make sense? Yes, there are about $14 \cdot 28.35 \approx 400$ grams in the box of cereal, and it did cost about ¾ of 400 cents.

Example:

My iTunes music library has about 9119 songs, for a total of 32.42 GB. Unfortunately, my iPod has a usable capacity of 27.8 GB. About how many of my songs can I fit on my iPod?

The iPod can hold about $\frac{27.8}{32.42} \approx 0.8575$ of my music library. Of my 9119 songs, then, it can hold about $0.8575 \cdot 9119 \approx 7819$ songs. Check: I was expecting to be able to fit most but not all my songs on my iPod, so this makes sense. (So I'll have to decide which 1300 songs I don't want on my iPod. I need a bigger iPod.)

About how many GB is each song? I want GB per song, so the units tell me to divide 32.42 by 9119 – I get about 0.0036 GB per song. (This would be more commonly reported as about 3.6 MB per song.)

Practice problems

1. Gasoline costs $(2.25, 2.75, 3.25, 3.50) per gallon. If you drive (12, 000, 15,000, 18,000) miles each year, and your car gets (23, 25, 28) miles per gallon, how much will it cost you to buy your gasoline each year?

Set up first with just the units; we want to end up with dollars per year:

$$\frac{dollars}{gallon} \cdot \frac{gallons}{mile} \cdot \frac{miles}{year} = \frac{dollars}{year}$$

Note that we needed to use gallons per mile in this expression to make the units "cancel" properly. Now we are ready to put in our numbers:

$$\frac{2.25\ dollars}{1\ gallon} \cdot \frac{1\ gallons}{28\ mile} \cdot \frac{15{,}000\ miles}{1\, year} = 1205.36 \frac{dollars}{year}$$

According to our calculations, you will spend about $1200 on gasoline each year.

Reality Check: Note that each of the original numbers is an estimate of an average – our answer will also only be an estimate. If gas prices change (should I say "when?"), or if we change how many miles we drive, this estimate could be way off.

2. A driver travels (n) hours at an average speed of (m) miles per hour. How many miles did she drive?

Set up with just the units first – we know hours and miles/hour, we want miles:

$$hours \cdot \frac{miles}{hour} = miles$$

3. A driver travels (n) miles at an average speed of (m) miles per hour. How many hours did her trip take?

Set up with just the units first – we know miles and miles/hour, we want hours:

$$miles \cdot \frac{hour}{miles} = hours$$

4. A driver drives (m) miles, taking (n) hours. What was her average speed?

Set up with just the units first – we know miles and hours, we want miles/hour (this is an easy one):

$$\frac{miles}{hour} = \frac{miles}{hour}$$