

# Arithmetic with Fractions

## Why do we need to be able to do this?

Why can't I just use my calculator?

It's often quicker to do arithmetic with fractions in your head than it is to find your calculator.

Being able at least estimate the answer in your head lets you check to see whether you did your calculator work correctly.

Many fractions don't have nice looking decimal forms, so the fraction is a simpler answer. A fraction also shows a proportion directly, and those are harder to see with decimal numbers.

Understanding how arithmetic works with fractions will help you when you're working with complicated algebraic expressions later.

Your teachers will expect you to be able to do fairly simple arithmetic with fractions quickly and automatically, and with no calculator help.

## What should you be able to do?

**Express a fraction in its simplest form:** Commonly known as “reducing” a fraction, this means writing the fraction so there are no common factors between its numerator and its denominator.

Example: Write the fraction  $35/105$  in its simplest form.

In order to see what factors might be common to the numerator and denominator, we need to factor each one. Finding the prime factorization is a good place to start:

$$\frac{35}{105} = \frac{5 \cdot 7}{3 \cdot 5 \cdot 7}$$

Note that there is a 5 in both the numerator and denominator, and there is a 7 in both the numerator and denominator. We can rewrite the fraction as:

$$\frac{35}{105} = \frac{5 \cdot 7}{3 \cdot 5 \cdot 7} = \frac{1 \cdot 5 \cdot 7}{3 \cdot 5 \cdot 7} = \frac{1}{3} \cdot \mathbf{1} \cdot \mathbf{1} = \frac{1}{3}$$

The  $\frac{5}{5}$  and the  $\frac{7}{7}$  are both just ways to write **1**. So  $\frac{35}{105}$  is really  $\frac{1}{3}$ , multiplied by **1** two times. Its simplest form is  $\frac{1}{3}$ .

**Express fractions with common denominators:** For much of the arithmetic you will do on fractions, you need to find a common denominator. That means finding some number that is a multiple of all of the denominators you start with.

Example: Find a common denominator for  $\frac{7}{12}$  and  $\frac{5}{8}$ , and express each fraction over that common denominator.

We need a multiple of both 12 and 8. One way to quickly find a common denominator is to simply multiply the denominators – in this case, a common denominator is  $12 \times 8 = 96$ .

To express a fraction over a common denominator, multiply it by **1** (so it doesn't change its value). The way you write the **1** is missing factor/ same missing factor.

$$\frac{7}{12} \cdot \frac{8}{8} = \frac{56}{96};$$
$$\frac{5}{8} \cdot \frac{12}{12} = \frac{60}{96}$$

**Find the prime factorization of two- and three-digit numbers.** To find common denominators, you will want to be able to quickly factor two-and three-digit numbers. Here are some strategies

1. Have the multiplication table memorized – then you can quickly recognize 56 as  $7 \times 8$ .
2. If you need to factor a number you don't know much about, you can divide it in turn by smaller numbers to look for factors. That sounds like a lot of work, but it doesn't need to be. You only have to try prime numbers, less than the number's square root. (Do you have to know the square root of the number? No, but you should be able to know what it's between.) And if you find a factor, you can look at the smaller, probably more familiar numbers. I find it most useful to start with the smallest primes and work my way up. And, yes, I do use a calculator sometimes at this point.

Example: Find the prime factorization of 132.

I can see that 2 is a factor, since 132 ends in 2. I'll divide that out to get  $132 = 2 \times 66$ . Now I can see that  $66 = 6 \times 11$  (multiplication tables). So  $132 = 2 \times 6 \times 11$ . These aren't all prime numbers, so let me factor the 6 a bit further.

$$132 = 2 \times 2 \times 3 \times 11, \text{ or } 2^2 \times 3 \times 11.$$

Example: Find the prime factorization of 193.

I don't know much about this number, so I'll just start dividing by prime numbers. I know I don't have to go past 14 (because  $14^2$  is 196, so this number's square root is less than 14).

I can see it isn't divisible by 2, since it ends in 3, an odd number.

I can also see that it isn't divisible by 3, since its digit sum is not divisible by 3 (a neat trick). If you aren't ready to use that trick, you can divide (sure, with your calculator) to see if you get a whole number.

I can see it's not divisible by 5, since the ones digit is not a 5 or a 0.

For 7, I have to divide: No, I don't get a whole number.

For 11, I have to divide: No, I don't get a whole number.

For 13, I have to divide: No, I don't get a whole number.

And that's it – there are no more primes less than 14, so I can stop. I didn't find any factors, which means that 193 is prime. Its prime factorization is  $193 = 193$ .

**Find (and use) the Least Common Denominator (LCD):** It is usually more convenient in the long run to find the Least Common Denominator, which is the smallest number that is a multiple of all the denominators.

Example: Find the Least Common Denominator for  $7/12$  and  $5/8$ , and express each fraction over that common denominator.

Sometimes you can just “see” what the LCD is – in this case, it is 24.

But if you can't see it at a glance, you can use the prime factorizations of the denominators to find the LCD. The LCD will have each prime factor you see, to the greatest power you see. In this case:

$$12 = 2^2 \cdot 3 \quad \text{and} \quad 8 = 2^3,$$

so we know that the LCD will be  $2^3 \cdot 3 = 24$ .

$$\frac{7}{12} \cdot \frac{2}{2} = \frac{14}{24};$$
$$\frac{5}{8} \cdot \frac{3}{3} = \frac{15}{24}$$

**Add or subtract fractions:** When adding or subtracting fractions, you need to express them with the same denominator. That means finding a common denominator, re-expressing each fraction over this common denominator (by multiplying by one), doing

the addition or subtraction (adding or subtracting the re-expressed numerators), and putting your answer in simplest form.

$$\text{Example: } \frac{3}{5} - \frac{2}{7} = \frac{3}{5} \cdot \frac{7}{7} - \frac{2}{7} \cdot \frac{5}{5} = \frac{21}{35} - \frac{10}{35} = \frac{21-10}{35} = \frac{11}{35}.$$

**Multiply fractions:** This is much easier than adding or subtracting – you can simply multiply numerators and denominators straight across.

$$\text{Example: } \frac{4}{15} \cdot \frac{2}{3} = \frac{8}{45}.$$

**Divide fractions:** This is that old “invert-and-multiply.” But which one do you invert? Remember that dividing by a fraction is the same as multiplying by its reciprocal. (Think about dividing by two being the same as multiplying by  $\frac{1}{2}$ , multiplying by  $\frac{1}{10}$  being the same as dividing by ten.) So the divisor, the fraction written after the division symbol or in the denominator of the compound fraction, is the one to invert.

$$\text{Example: } \frac{4}{15} \div \frac{2}{3} = \frac{4}{15} \cdot \frac{3}{2} = \frac{12}{30} = \frac{2}{5}.$$

**Estimate size:** You can often make a crude estimate of the answer to an arithmetic problem involving fractions by thinking about the size of the parts. You should be able to estimate whether a fraction is closer to zero, to  $\frac{1}{2}$ , or to 1, for example. You can think in terms of wholes, halves, thirds, quarters, and tenths – these are the fractions we are most used to.

Example: Did we invert the right fraction in the example above? Is the answer to  $\frac{4}{15} \div \frac{2}{3}$  really  $\frac{2}{5}$ ? Or should it be  $\frac{5}{2}$  instead?

Remember that dividing means the same thing whether it's whole numbers or fractions – how many  $\frac{2}{3}$ s will fit in  $\frac{4}{15}$ ?  $\frac{4}{15}$  is about  $\frac{1}{4}$ , so  $\frac{2}{3}$  is bigger than  $\frac{4}{15}$ , so less than one  $\frac{2}{3}$  will fit inside  $\frac{4}{15}$ . Our answer should be less than one. So  $\frac{2}{5}$  makes more sense than  $\frac{5}{2}$ .